obtained in Ref. 1 is not only an approximation to that found by Beck, but it is also an exact solution for a theoretically massless structure. As far as the author is aware, this fact has not been referred to explicitly in the literature. Thus, in what follows, a vibration stability analysis will be carried out explicitly for a massless Beck column with a single concentrated point mass m at the free top where the tangential force P is acting.

Exact Solution

Following Fig. 1, the equilibrium condition $\Sigma M_{(l)} = 0$ gives

$$M + P\cos\varphi_l(\omega_l - \omega) - m\ddot{\omega}_l(l-x) - P\sin\varphi_l(l-x) = 0 \quad (1)$$

and since

$$\cos\varphi_l \cong 1, \qquad \sin\varphi_l = \omega'_l, \qquad \text{and} \qquad M \cong -EI\omega''$$

the differential equation of the problem is

$$EI\omega'' + P\omega - P\omega_\rho + (P\omega'_l + m\ddot{\omega}_l)(l-x) = 0 \quad (2)$$

where $(\,') = [d(\,)/dx]$, $(\,\dot{}\,) = [d(\,)/dt]$, and t is the time. This partial differential equation can be reduced to an ordinary differential equation in the usual way by setting[2]

$$\omega = W_{(x)} e^{i\lambda t} \quad (3)$$

where λ is the frequency. The reduced differential equation is thus

$$EIW'' + PW - PW_l + P(W'_l - m\lambda^2 W_l)(l-x) = 0 \quad (4)$$

The general solution of this differential equation is

$$W = A\sin\nu x + B\cos\nu x + W_l - (W'_l - \lambda^2 \frac{m}{P} W_l)(l-x) \quad (5)$$

where $\nu = \sqrt{P/EI}$ and A and B are constants. With the boundary conditions of the problem

$$W = W' = 0 \qquad \text{for} \qquad x = 0$$

and

$$W = W_l \quad \text{and} \quad W' = W'_l \quad \text{for} \quad x = l$$

the following four equations for A, B, W_l, *and* W'_l are obtained

$$B + [1 + \lambda^2(m/P)l]W_l - lW'_l = 0 \quad (6)$$

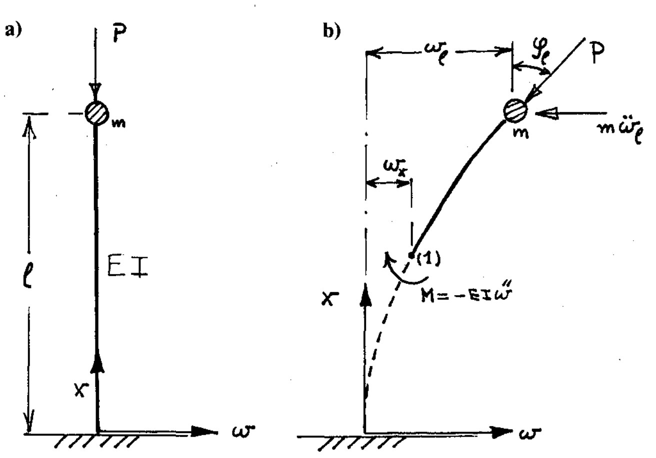

Fig. 1 Geometry and notation of the Beck column in a) the fundamental state, and b) the buckled state.

$$\nu A - \lambda^2(m/P)W_l + W'_l = 0 \quad (7)$$

$$A\sin\nu_l + B\cos\nu_l = 0 \quad (8)$$

$$\nu A\cos\nu l - \nu B\sin\nu l - \lambda^2(m/P)W_l = 0 \quad (9)$$

This is a system of homogeneous linear algebraic equations which has a nontrivial solution only if the determinant of the coefficients vanishes. This determinant in the case of a nonconservative system is termed the frequency determinant. Expanding the frequency determinant we obtain the secular frequency equation

$$3\lambda^2\nu l(\sin\nu l - \nu l\cos\nu l) - \nu^4 l^4\Omega^2 = 0 \quad (10a)$$

$$\sin\nu l - \nu l\cos\nu l = D \quad (10b)$$

where $\Omega = \sqrt{3EI/ml^3}$. Since stability in the sense of Liapanov is only possible when none of the roots of the frequency equation has a positive, real part,[2] we are led to the following instability condition:

$$D = \sin\nu l - \nu l\cos\nu l = 0 \quad (11)$$

The solution of this equation is

$$\nu l = (P/EI)^{1/2} l = 4.493 \quad (12)$$

The buckling load is, therefore,

$$P^c = (4.49)^2 EI/l^2 = 20.19\, EI/l^2 \quad (13)$$

This buckling load is identical with that obtained on the grounds of purely static consideration by Ingerle and the present author.[1]

Conclusions

A previous result for the buckling load of the Beck problem which was obtained on the grounds of purely statical consideration is reconsidered. It is shown that this result not only constitutes an approximation, but also represents an exact solution under certain circumstances.

Acknowledgment

The author is deeply indebted to S. Athel for stimulating this research and for his continuous interest and helpful criticism.

References

[1] El Naschie, M.S., "Postcritical Behavior of the Beck Problem," *Journal of Sound and Vibration,* Vol. 38, 1976, pp. 341-344.

[2] Beck, M., "Die Knicklast des Einseitig Eingespannten, Tangential Gedruckten Stabes," Zeitschrift für Mat matik angwandete Ratirmatik und physik), Vol. 3, 1952, pp. 225-228.

Mass Transfer in a Supersonic Near Wake

G. B. Diep* and P. Magret†
Laboratoire d'Aerothermique du C.N.R.S.
Meudon, France

Introduction

THE subject of base flow in laminar supersonic conditions has been covered by many investigations.[1-6] Weiss,[7] in

Received Nov. 8, 1976; revision received April 4, 1977.
Index categories: Supersonic and Hypersonic Flows; Jets, Wakes, and Viscid-Inviscid Flow Interactions.

*Maître de Recherche.
†Ingénieur de Recherche.

applying his flow model, is able to deal with the entire near wake by intercoupling the three essential base flow regions: the outer region, the shear layer, and the recirculation region, for which he solves the incompressible Navier-Stokes equations. This model provides a satisfactory comparison between theoretical results concerning the dynamic and thermal fields and the experiments.[8] Similarly, Tran Cam Thuy[9] suggests a new digital processing of the recirculation region equations. Concurrent with these investigations, tests were conducted with the aim of reducing the heat-transfer coefficients in the near wake by the injection of a foreign gas. In the case of an injection outside the bubble zone, detailed investigations called for measurements of the concentrations of the injected gas arriving downstream from the base. In this context, the purpose of this study was to modify a theoretical model already employed for near wake problems, so as to describe the mechanism of diffusion and to compare these results with the diffusion field introduced experimentally into a base flow.

Diffusion Phenomena in Weiss Theory

In the case of base flows with moderate Reynolds numbers, three regions can be distinguished in the near wake. The first region (outer region) corresponds to the expansion of the boundary layer followed by supersonic recompression in the vicinity of the reattachment point. In the initial stage, it is assumed that the boundary layer expands to the base pressure, and the velocity profile after this expansion is derived simply from the initial profile by a rotation through an angle equal to that of the Prandtl-Meyer expansion. The momentum thickness and concentration thickness are conserved, and these values serve to define the starting profiles of the shear layer.

In the second region (shear layer), the limiting conditions at the dividing streamline and the Oseen linearization method make it possible to derive analytic expressions of the velocity and concentration at any point of this region.

In the last region (recirculation), the Navier-Stokes equations in incompressible flow, after differentiation, give respectively, the stream function and vorticity. Each of these regions can be computed with the help of boundary conditions created by the immediately adjacent region. It is necessary to employ an iterative coupling procedure between these regions to reach a convergent solution. A computation program based on this model was written in Fortran IV, and the computation requires 2 min on an IBM 370-168 computer.

Experimental System

Tests were performed in a continuous open wind tunnel with the following main characteristics: 1) Mach number, 4.92; 2) stagnation temperature, 453 K; 3) stagnation pressure, 8.53×10^6 Pa; 4) Reynolds number per meter, 1.3×10^7.

The model is a rectangular plate in which a 9.5-mm-high step H has been cut out 55 mm from the leading edge. Upstream from the step, the tracer injection system consists of a cavity closed by a centered bronze plate. The argon or helium tracer injection conditions (flowrate, temperature, pressure) are monitored. Gas samples, taken through an 0.3-mm-o. d. pitot tube, are sent to pressure sensors or a mass spectrometer. Introduction of the test sample into the chamber, which is facilited by auxiliary pumping, is effected by means of a valve that controls the pressure in the chamber at a given value, through the intermediary of a Bayard-Alpert gage. The chamber is placed under vacuum between samplings. This experimental method, already described by Collins et al.,[10] does not produce selective pumping, as observed by these authors. In the case of continuous pumping of substances of comparable atomic weights, we failed to detect deviations in the measured concentrations. Finally, it is necessary to characterize the intensity of the tracer gas injection, which depends on its momentum C_q, which is defined as the ratio $C_q = \rho_p v_p / \rho_\infty v_\infty$, p indicating the injection conditions.

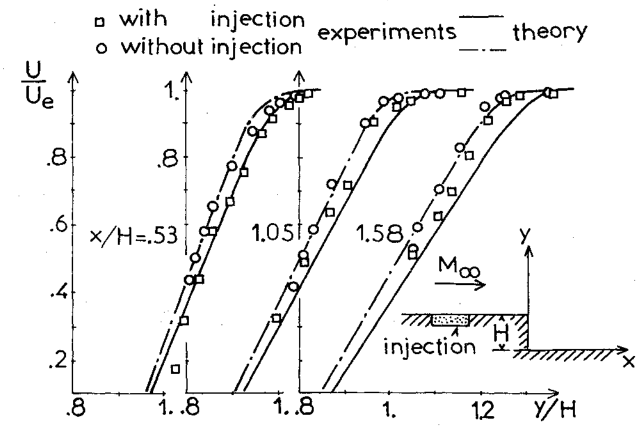

Fig. 1 Theoretical and experimental velocity profiles.

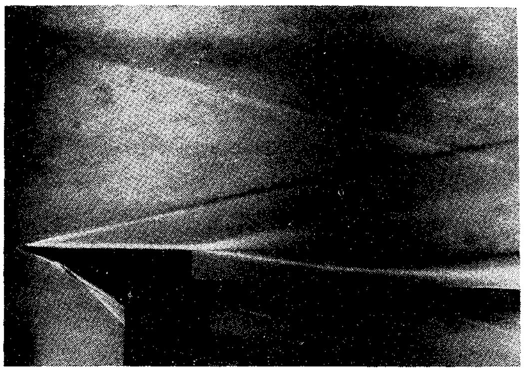

Fig. 2 Schlieren photograph of the flowfield.

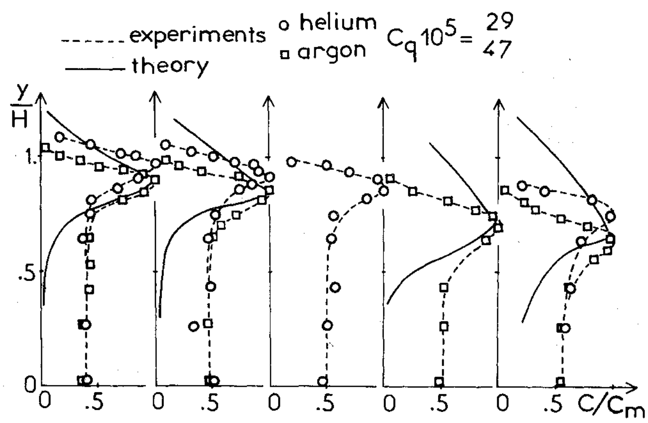

Fig. 3 Theoretical and experimental concentration profiles for helium and argon injections.

Comparison of Results

In Fig. 1 are plotted the theoretical and experimental velocity profiles in the shear region, for a given $C_q = 4.7 \times 10^{-4}$. It can be seen that the dividing streamlines does not pass exactly through the edge of the step, whereas our computation hypotheses assumed that this line was straight from the singular point and inclined by the Prandtl-Meyer expansion angle.

Figure 2 shows that the recompression region is rather poorly defined, and that the position of the reattachment point is influenced by an interaction tending to push it downstream: repercussions on the entire dividing streamline cannot be excluded. The concentration profiles that are observed (Fig. 3) indicate the significant presence of tracer at the base of the step. The concentration maxima C_m are located in the immediate vicinity of the dividing streamline. We also employed helium (the Schmidt and Prandtl numbers are comparable, despite differences in density). Not withstanding the differences between maximum concentration of the two gases, the concentration profiles tend toward identical values at the wall. But a small significant gap exists between the maximum concentration ordinates obtained for the two injections: helium appears to exhibit greater diffusion toward the outer region. However, the stagnation region – incompressible flow region – practically is not modified by injection. The boundary of this region is naturally very close to the experimental dividing streamline.

Conclusions

Diffusion phenomena in a cavity were investigated by adopting a computation scheme designed for thermal problems. The Navier-Stokes equations with mass transfer were solved in two coupled regions. This model led to the development of a computation program corresponding to the injection of a foreign gas upstream from the near wake, this injection only slightly modifying the dynamic field. The experiments made it possible to determine the existence of dynamic and concentration fields in the base flow.

Comparison of theoretical and experimental results gave rise to the following conclusions:

1) The results are in good agreement with respect to the dynamic field. The gaps that subsist derive from the hypotheses concerning the dividing streamline – boundary between the two regions investigated – made in the simplified model.

2) As for the diffusion field, which quickly reaches a state of equilibrium, a gap exists concerning the mass concentration value at the horizontal wall. The closely comparable results obtained for this value with helium and argon, which have widely differing molecular weights, diffusivity and conductivity, exclude the influence of gravity and thermodiffusion, which are present in the dividing streamline.

References

[1]Chapman, D. R., Kuehn, D. M., and Larson, H. K., "Investigation of Separated Flows in Supersonic and Subsonic Streams with Emphasis on the Effects of Transition," NACA TN 3869, 1957.

[2]Korst, H. H., "Theory for Base Pressure in Transonic and Supersonic Flows," *Journal of Applied Mechanics,* Vol. 78, Dec. 1956, pp. 593-600.

[3]Denison, M. R. and Baum, E., "Compressible Free Shear Layer with Initial Thickness," *AIAA Journal*, Vol. 1, Feb. 1963, pp. 342-349.

[4]Hubbartt, J. E., "Approximate Solution of Laminar Free Layers with Initial Thickness," *AIAA Journal*, Vol. 3, Aug. 1965, pp. 1538-1540.

[5]Lees, L. and Reeves, B. L., "Supersonic Separated and Reattaching Laminar Flows. General Theory and Application to Boundary Layer Shock Wave Interaction," *AIAA Journal*, Vol. 2, Nov. 1964, pp. 1907-1920.

[6]Weinbaum, S., "The Rapid Expansion of a Supersonic Boundary Layer and its Application to the Near Wake," *AIAA Journal*, Vol. 4, Feb. 1966, pp. 217-226.

[7]Weiss, R. F., "A New Theoretical Solution of the Laminar Hypersonic Near Wake," *AIAA Journal*, Vol. 5, Dec. 1967, pp. 2142-2149.

[8]Todisco, A. and Pallone, A., "Near Wake Flow Field Measurements," *AIAA Journal*, Vol. 3, Nov. 1965, pp. 2075-2080.

[9]Tran Cam Thuy, "Résolution numérique des équations de Navier-Stokes," Thése de Docteur-Ingénieur présentée à la Faculté des Sciences de Paris, Déc. 1970.

[10]Collins, D. J., Lees, L. and Roshko, A., "Near Wake of a Hypersonic Blunt Body with Mass Addition," *AIAA Journal*, Vol. 8, May 1970, pp. 833-842.

A Near-Optimal Control Law for Pursuit-Evasion Problems Between Two Spacecraft

G. M. Anderson* and G. D. Bohn†
Air Force Institute of Technology,
Wright-Patterson Air Force Base, Ohio

Introduction

IN Refs. 1 and 2, a general method for generating near optimal feedback solutions to nonlinear zero-sum differential games is presented which allows a player using this method to take advantage of any nonoptimal play of his opponent. In employing this method, a player periodically updates, to first order, his solution to the two-point boundary-value problem (TPBVP) obtained using the necessary conditions for a differential game saddle point solution. At an updating time t_i, this update is accomplished by adjusting the costate vector $\lambda(t_i)$ according to

$$\delta\lambda(t_i) = S(t_i)\delta x(t_i) \tag{1}$$

where $\delta x(t_i)$ is the difference between the actual system state and a reference state indicated by a TPBVP solution. In Ref. 1, $S(t_i)$ is generated by the backward integration of a matrix Riccati differential equation from t_f to t_i, whereas in Ref. 2 it is obtained using the transition matrices for the linearized TPBVP.

The examples presented in these references all are relatively simple with low state dimension. The purpose of this Note is to demonstrate the capability of this method for generating near optimal feedback controls for a more realistic pursuit-evasion problem between two maneuvering space vehicles. The pursuing vehicle employs the near-optimal scheme, whereas the evading vehicle plays various nonoptimal strategies. The payoff is the final range between the two vehicles, the final time is left free, and the game ends when the range rate between the vehicles goes to zero. Coplanar problems are investigated using both the matrix Riccati and transition matrix updating techniques, whereas noncoplanar problems are solved using only the matrix Riccati method.

Problem Statement

The equations describing the motion of a thrusting space vehicle in an inverse-square gravitational field are

$$\dot{r} = V_r \quad \dot{\theta} = V_\theta / r \quad \dot{\phi} = V_\phi / (r \sin\theta)$$

$$\dot{V}_r = (V_\theta^2 + V_\phi^2)/r - 1/r^2 + (F/m)\sin\alpha_2$$

$$\dot{V}_\theta = (V_\phi^2 \cot\theta - V_r V_\theta)/r + (F/m)\cos\alpha_1 \cos\alpha_2$$

$$\dot{V}_\phi = -(V_r V_\phi + V_\theta V_\phi \cot\theta)/r + (F/m)\sin\alpha_1 \cos\alpha_2 \tag{2}$$

where Earth canonical units are used with an Earth-centered spherical coordinate system. The controls are the angles α_1 and α_2, which define the thrust direction. The normalized

Presented as Paper 76-794 at the AIAA/AAS Astrodynamics Conference, San Diego, Calif., Aug. 18-20, 1976; submitted Sept. 15, 1976; revision received May 5, 1977.

Index categories: Analytical and Numerical Methods; Spacecraft Dynamics and Control; Spacecraft Navigation, Guidance, and Flight-Path Control.

*Professor, Department of Aeronautics and Astronautics; presently Senior Principal Engineer, ORINCON Corp., La Jolla, Calif. Member AIAA.

†Graduate Student, Department of Aeronautics and Astronautics.